Finite-size Lyapunov exponent for Levy processes

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The finite-size Lyapunov exponent (FSLE) is the exponential rate at which two particles separate from a distance of r to $a\times r$ $(a>1)$ and provides a measure of dispersive mixing in chaotic systems. It is shown analytically that for particle trajectories governed by symmetric α-stable Levy motion, the FSLE is proportional to the diffusion coefficient and inversely proportional to r^{α}. This power law provides an easy method to determine the parameters for Levy processes and hence has applications to superdiffusion in the atmospheric, oceanic, and terrestrial sciences.

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The Lyapunov exponent is commonly used to quantify mixing in fluid mechanics and is defined as the exponential rate of separation, averaged over infinite time, of fluid parcels initially separated infinitesimally. In other words, the Lyapunov exponent represents the average rate of exponential divergence of two infinitesimally close trajectories. Consider $\mathbf{X}_1(t)$ and $\mathbf{X}_2(t)=\mathbf{X}_1(t)+\mathbf{R}(t)$ as two particle trajectories separated initially by a distance $R(0)$. The Lyapunov exponent is defined by

$$\lambda = \lim_{t\to\infty}\lim_{R(0)\to 0}\frac{1}{t}\ln\frac{R(t)}{R(0)}, \tag{1}$$

where $R(t)$ is the magnitude of $\mathbf{R}(t)$.

The infinite-time limit in Eq. (1) makes the Lyapunov exponent of limited practical use when dealing with experimental data. The second limit $[R(0)\to 0]$ makes it an even more difficult quantity to evaluate either experimentally or numerically. However, a generalization of the Lyapunov exponent, called the finite-size Lyapunov exponent (FSLE), has been proposed [1–3] to study the growth of noninfinitesimal perturbations (distance between trajectories) in dynamical systems. Recently the concept of a FSLE has been applied to study dispersion in turbulent flow fields [1–6] and dispersion in porous systems [4]. It has been applied to study the relative dispersion on the surface of the Gulf of Mexico [7] and to identify the mixing structures in the Mediterranean Sea [8]. Lacorata *et al.* [9] have employed the FSLE to study the transport of balloons in the lower stratosphere.

A rigorous statistical mechanical definition of the FSLE was recently presented by Kleinfelter *et al.* [4] that takes into account such phenomena as recirculation, nonstationarity, and periodic processes. If N is the total number of particles in a completely expansive system and the joint probability measure that particles i and j have separation in $(r_{i,j}, r_{i,j}+dr_{i,j})$ at time τ and separation in $(r'_{i,j}, r'_{i,j}+dr'_{i,j})$ at time $t+\tau$ is denoted by $f(r_{1,2}(\tau), r_{1,3}(\tau),\ldots, r_{N-1,N}(\tau), r'_{1,2}(t+\tau), r'_{1,3}(t+\tau), r'_{N-1,N}(t+\tau))dr_{1,2}dr'_{1,2}\ldots dr_{N-1,N}dr'_{N-1,N}$ in a $[N^2(N-1)^2]$-dimensional pair separation space $\Omega(t,\tau)=\Omega(\tau)\times\Omega(t+\tau)$, then the conditional probability of any two particles being separated by a distance $a\times x$ for the first time at $t+\tau$, given they were separated by x at time τ, is expressed as

$$G_\tau(x,ax;t)=\frac{1}{N(N-1)}\sum_{i\neq j}\frac{\langle\langle\delta(x-r_{i,j}(\tau))\delta(ax-r_{i,j}(\tau+t))\rangle\rangle}{\langle\langle\delta(x-r_{i,j}(\tau))\rangle\rangle}, \tag{2}$$

where δ is the usual Dirac delta function and $\langle\langle\cdot\rangle\rangle$ indicates integration over $\Omega(t,\tau)=\Omega(\tau)\times\Omega(t+\tau)$ with respect to the joint probability density defined above. The average conditional probability over all pairs of particles to reach separation of $a\times r$ for the first time at t, given that they were separated by r at time zero, can be found by integrating Eq. (2) for all possible values of τ as

$$G(r,ar;t)=\int_0^\infty G_\tau(r,ar;t)d\tau. \tag{3}$$

If r is the initial separation between two trajectories and $T_a(r)$ is the mean time it takes the separation to grow by a factor $a>1$, then one can write

$$T_a(r)=E[G(r,ar;t)]=\int_0^\infty tG(r,ar;t)dt. \tag{4}$$

The FSLE of a diverging system is defined as

$$\lambda_a(r)=\frac{1}{T_a(r)}\ln a. \tag{5}$$

In contrast to the earlier definitions, the FSLE in [4] was defined statistically to account for flows where particles may continuously resample separations. Moreover, the FSLE was defined to be a continuous function of the initial separation which paves the way for a more detailed theoretical study.

The physical concept that forms the basis of the FSLE is that the separation r between trajectories represents the scale at which one observes the system and hence the FSLE allows one to quantify mixing at different length scales. The FSLE is a natural tool for analyzing Lagrangian data sets. By means of particle tracking algorithms, one can select nearby particles separated by a given distance, say r, and then follow them measuring the time $T_a(r)$ it takes for the separation to grow to $a \times r$. In the Fickian limit, it has been argued that the FSLE will be inversely proportional to the square of initial separation and directly proportional to the classical diffusion coefficient [3].

Most recently, the FSLE was employed to investigate anomalous dispersion in processes characterized by Levy motion. Levy motion is a superset of Brownian motion where trajectory increments are quantified statistically by an α-stable distribution [10]. Based on numerical experiments, Kleinfelter *et al.* [4] suggested that for α-stable Levy processes, the relationship

$$\lambda_a(r) \sim C_a r^{-\alpha} \tag{6}$$

holds with the dependence of a entirely in the coefficient of proportionality. Should this result be exact, it would provide a quick and easy way to determine the stability parameter α from data that display a Levy-process behavior. The motivation for this Brief Report is to establish Eq. (6) using theoretical means and to give an analytical expression for the coefficient of proportionality, C_a. The reader should keep in mind that a power law relation, such as displayed in (6), does not imply the process is Levy, but if the process is Levy, then it will display this power law.

As indicated above, Levy motion is a generalization of the concept of Brownian motion to processes with infinite variance. Classical Brownian diffusion is characterized by a Gaussian distribution. In Levy motion, long flights are interspersed with shorter jumps, resulting in infinite variance and fractal paths. The transition density for increments is given by an α-stable distribution [10,11]. Gaussian distributions and symmetric α-stable distributions look very much alike except that the latter is characterized by a heavier tail. Whereas Gaussian distributions decay quickly, α-stable distributions decrease as $1/x^{1+\alpha}$, where α lies between 0 and 2. A famous example is the Cauchy-Lorentz density, which is proportional to $1/(c+x^2)$, where c is a positive constant. This distribution corresponds to $\alpha=1$ and arises in many physical situations. The Gaussian distribution corresponds to the case where $\alpha=2$.

Levy proved a generalization of DeMoive's central limit theorem by removing the assumption of finite variance on iid random variables X_i [12,13]:

$$\lim_{n\to\infty} \frac{X_1 + X_2 + \cdots + X_n - n\mu}{\sigma n^{1/\alpha}} = Y \sim S_\alpha(\sigma=1,\beta,\mu=0), \tag{7}$$

which states that the centered and normalized sum of iid random variables converges in distribution to an "α-stable" variable with index of stability $0<\alpha\leq 2$, skewness coefficient $-1\leq\beta\leq 1$, shift parameter $\mu=0$, and spread parameter $\sigma=1$.

By envisioning the sum in Eq. (7) as a random walk with the number of steps, n, equal to the total time divided by the time per move, Δt, we can recast the generalized central limit theorem in the form

$$\lim_{t/\Delta t\to\infty} \frac{X_1 + X_2 + \cdots + X_{t/\Delta t} - vt}{(Bt)^{1/\alpha}} = Y \sim S_\alpha(\sigma=1,\beta,\mu=0), \tag{8}$$

where $v=\mu/\Delta t$ and $B=\sigma^\alpha/\Delta t$. The parameters μ and σ appearing in the definition of v and B are the shift and spread parameters for each iid random variable X_i. A stochastic process $\{X(t):t\geq 0\}$ is called an α-stable Levy motion if $\{X(t):t\geq 0\}$ has independent increments and $X(t)-X(s) \sim S_\alpha((t-s)^{1/\alpha},\beta,0)$ for any $0\leq s<t<\infty$. Since most stable densities cannot be written in real space, they are typically expressed in terms of their Fourier transforms (characteristic functions) as [10]

$$P(k,t) = \exp\left[-ikX_0 - \left\{Bt|k|^\alpha\left(1 - i\beta\,\mathrm{sgn}(k)\tan\frac{\pi\alpha}{2}\right) + ikvt\right\}\right], \tag{9}$$

where $X_0=X(0)$ is the initial position of the walk and $\mathrm{sgn}(k)$ is 1 if $k>0$ and -1 if $k<0$. The positive number $(Bt)^{1/\alpha}$ in Eq. (8) indicates that the stable density is invariant upon scaling by $t^{1/\alpha}$. Using a well-known property of the characteristic function—namely, $\langle X^n(t)\rangle = i^n[d^nP(k,t)/dk^n]_{k=0}$—one can easily show that for all $m>\alpha$, $\langle X^m\rangle=\infty$. Hence for any $\alpha<2$, α-stable densities have infinite variance. However, a finite sampling of the density will yield a finite sample variance. Since the densities are scale invariant with $t^{1/\alpha}$, the sample variance will grow proportional to $t^{2/\alpha}$—i.e., always equal to or faster than Fickian growth. For cases where the distribution is not skewed ($\beta=0$), Eq. (9) reduces to

$$P(k,t) = \exp[-ikX_0 - \{Bt|k|^\alpha + ikvt\}]. \tag{10}$$

The value of α in Eqs. (9) or (10) ggdetermines how non-Gaussian a particular density becomes. As the value of α decreases from a maximum of 2, the tail of the probability density becomes more pronounced. These interesting properties of the α-stable variables have found widespread use in a variety of fields, such as in the study of microbial dynamics [14,15], transport in porous media [16–18], anomalous dispersion [19], and superdiffusion in turbulence [20,21].

The Fokker-Planck equation for α-stable Levy motion is fractional [22] and is given in one dimension by

$$\frac{\partial P(x,t)}{\partial t} = D\frac{\partial^\alpha P(x,t)}{\partial x^\alpha}, \tag{11}$$

where $D=B/|\cos(\pi\alpha/2)|$ and $P(x,t=0)=\delta(x-x_0)$. In real space the fractional derivatives are integro-differential operators, and hence Eq. (11) describes a spatially nonlocal process. A spatial fractional derivative therefore describes particles that move with long-range spatial dependence or high-velocity variability [11]. If a one-dimensional random walk is considered as a motion on an infinite lattice, then a larger-order fractional derivative places more weight on nearer cells and probability of jumps to distant cells decrease very quickly with distance. Similarly, lower-order fractional

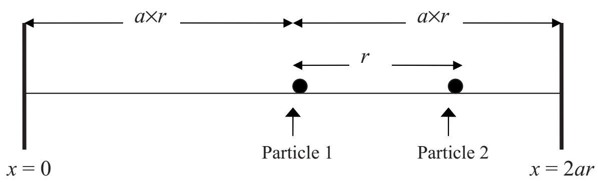

FIG. 1. Position of particles at $t=0$ and location of the passage planes.

derivatives place relatively less weight on nearer cells and probability of jumps to distant cells decrease slowly with distance.

By generating realizations of α-stable Levy motions, Kleinfelter *et al.* [4] suggested that the stability parameter might be found by measuring the slope of a log-log plot between the FSLE and the initial separation r of two trajectories. It was further suggested that the coefficient of proportionality, C_a, has a decreasing dependence on a as α decreases. However, the numerical results were for a fixed value of σ and hence a functional dependence of FSLE on the spread parameter was not discussed.

The FSLE is obtained experimentally by generating trajectories and then comparing pairs of trajectories. In essence, computation of the FSLE is directly linked to computation of $T_a(r)$ [see Eq. (4)] where $T_a(r)$, which is the average time it takes for two trajectories to reach a separation of $a\times r$, is called the a time. To develop an analytical expression for $T_a(r)$, consider two particles initially separated by a distance r undergoing α-stable Levy motion. Let particle 1 be initially located at a distance $x=a\times r$ from the origin (Fig. 1) and bounded by two first-passage planes at a distance of $a\times r$ from the particle.

Independent incremental changes after each time interval in the spatial location of particles 1 and 2—i.e., Δx_i^1 and Δx_i^2—are described by the same α-stable distribution $S_\alpha(\sigma,\beta=0,\mu=0)$. The particles will thus disperse in the one-dimensional medium and for some mean time $T_a(r)$ will reach a separation of $a\times r$. An intuitive way to understand $T_a(r)$ is to observe the system given in Fig. 1 from a reference frame that is attached to particle 1. After each time interval Δt, the whole system (including passage planes) will then move by an increment Δx_i^1. In this comoving frame, the only mobile entity depicted in Fig. 1 is particle 2. However, the incremental change in the location of particle 2 after each time interval is now given by the difference between Δx_i^2 and Δx_i^1. Because the difference between two α-stable Levy processes is also an α-stable Levy, it can be shown that the jump lengths $(\Delta x_i^2-\Delta x_i^1)$ for particle 2 in the comoving frame will have the distribution $S_\alpha(2^{1/\alpha}\sigma,\beta=0,\mu=0)$ [10]. The time $T_a(r)$ in the comoving frame can be defined as the expected time it takes a particle which is initially located at $x_0=ar+r$ to reach either of the passage planes located at $x=0$ and $x=2ar$. It is important to note that particle jumps in our moving system are governed by a modified α-stable distribution, the stability parameter of which is the same as that of the original distribution, but with the spread parameter modified by a factor of $2^{1/\alpha}$. The constants B and D, which are a function of σ, will be modified accordingly.

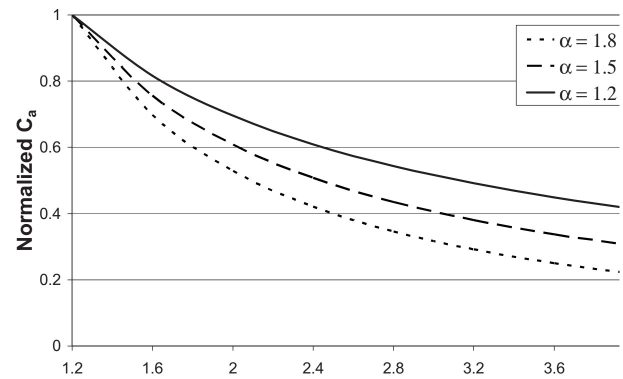

FIG. 2. Dependence of C_a on a for different values of α.

In the comoving frame, $T_a(r)$ is obtained by finding the mean first-passage time where the locations of passage planes are fixed at $x=0$ and $x=2ar$ and the initial position of the particle is at a distance of $ar+r$ from the left plane. The passage planes should be treated as absorbing, so the particle cannot reenter the system. An analysis of the mean first-passage time, based upon a related formulation, was presented by Gitterman [23]. He showed that a particle with an initial condition given by $P(x,t=0)=\delta(x-x_0)$ reaches one of the absorbing barriers with an expected time

$$T=\frac{2}{\pi D}\left(\frac{L}{\pi}\right)^\alpha\sum_{n=1}^{\infty}\frac{[1-(-1)^n]\sin(n\pi x_0/L)}{n^{1+\alpha}}, \tag{12}$$

where L is the distance between two passage planes. Substituting $L=2ar$ and $x_0=ar+r$ (as given in Fig. 1), Eq. (12) reduces to

$$T_a(r)=\frac{2^{2+\alpha}a^\alpha}{\pi^{1+\alpha}D}(r)^\alpha\sum_{m=0}^{\infty}\frac{(-1)^m\sin\left[(2m+1)\left(1+\frac{1}{a}\right)\frac{\pi}{2}\right]}{(2m+1)^{1+\alpha}}. \tag{13}$$

Substituting Eq. (13) into Eq. (5) yields the FSLE for α-stable Levy processes as

$$\lambda_a(r)=\frac{\pi^{1+\alpha}D\ln(a)}{2^{2+\alpha}a^\alpha P(a,\alpha)}(r)^{-\alpha}, \tag{14}$$

where

$$P(a,\alpha)=\sum_{m=0}^{\infty}\frac{(-1)^m\sin\left[(2m+1)\left(1+\frac{1}{a}\right)\frac{\pi}{2}\right]}{(2m+1)^{1+\alpha}}. \tag{15}$$

Equation (14) validates Eq. (6) which had been computationally obtained by Kleinfelter *et al.* [4]. The FSLE for an α-stable distribution is directly proportional to the diffusion coefficient and inversely proportional to the α power of the initial separation between particles. The diffusion coefficient $D=(2\sigma^\alpha)/(\Delta t|\cos(\pi\alpha/2)|)$ is a function of σ alone if α is fixed. Comparing Eq. (6) with Eq. (14) yields an expression for C_a as

$$C_a = \frac{\pi^{1+\alpha} D \ln(a)}{2^{2+\alpha} a^{\alpha} P(a,\alpha)}. \tag{16}$$

Figure 2 illustrates the dependence of C_a on a for different values of α. The vertical axis in Fig. 2 has been normalized by $C_{a=1.2}$ so that a visual comparison can be made between lines corresponding to different values of α. The plots were generated for a fixed value of σ and Δt. It is clear from the plot that as α decreases, the slope of the C_a vs a line becomes milder. In other words, there is a decreasing dependence of C_a on a as α decreases. This observation concurs with the numerical findings of Kleinfelter *et al.* [4].

The above analysis establishes a simple method for determining the stability parameter α of a Levy process by measuring the time $T_a(r)$ using any available particle-tracking algorithm. Equation (14) also paves the way to find the spread parameter σ for a Levy process. The constant of proportionality between $\lambda_a(r)$ and r in Eq. (14) is a function of σ only, once a is fixed and the stability index α is found from the slope of a log-log plot.

Often one is interested in computing the doubling time—i.e., $T_a(r)$ corresponding to $a=2$. For this special case, using regression, $P(2,\alpha)$ can be approximated by a second-degree polynomial as

$$P(2,\alpha) = -0.055\alpha^2 + 0.222\alpha + 0.45, \tag{17}$$

with a regression coefficient of 0.998. For the case when $(1+\alpha)/2$ is an integer (i.e., $\alpha=1$), the summation in Eq. (15) can be expressed analytically as $P(2,1)=\pi^2/16$ [24].

To summarize, an exact expression for the FSLE for an α-stable Levy process was obtained. The FSLE was shown to provide a simple method to compute the stability parameter α and the spread parameter σ from the data. The power-law relation between $\lambda_a(r)$ and r of Eq. (14), when plotted on a log-log scale, will yield α as the slope and will facilitate computation of σ by estimating the constant of proportionality.

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